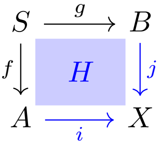
~~Towards a~~ Formalization of Wärn's zigzag construction

Vojtěch Štěpančík

Outline

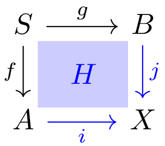
Pushout of $A \xleftarrow{f} S \xrightarrow{g} B$ in agda-unimath

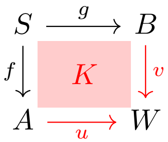
- A cocone (i, j, H) is a pushout



Pushout of $A \xleftarrow{f} S \xrightarrow{g} B$ in agda-unimath

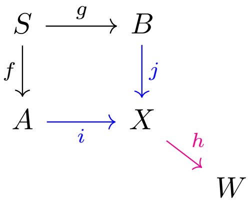
- A cocone (i, j, H) is a pushout if every cocone under the same span

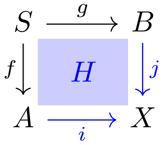


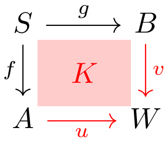


Pushout of $A \xleftarrow{f} S \xrightarrow{g} B$ in agda-unimath

- A cocone (i, j, H) is a pushout if every cocone under the same span induces a unique map $h : X \to W$







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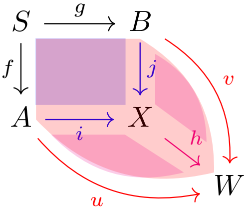

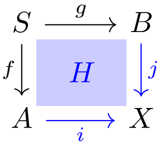

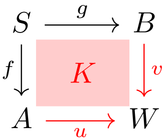

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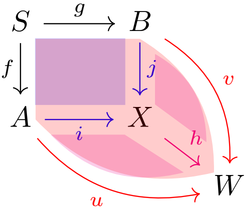
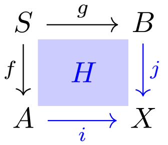
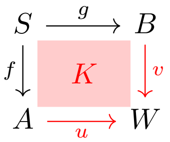


- $\leftrightarrow$ "the map from $(X \to W)$ to cocones with vertex W is an equivalence"





Describing identity types of pushouts

- Pushouts are a fundamental method for creating spaces by gluing

[1]Wärn. Path Spaces of Pushouts, 2023, https://dwarn.se/po-paths.pdf

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- Wärn described the "zigzag construction"[1] in 2023
- No formalization existed for almost two years, until now!

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Setting

- Book HoTT
- Built with agda-unimath[2], with plans to upstream

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Consequences of choosing agda-unimath

- Postulated pushouts
- Reusable code
- Performance matters

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Conventions

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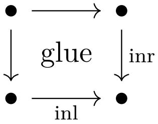

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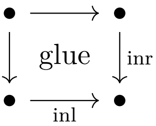
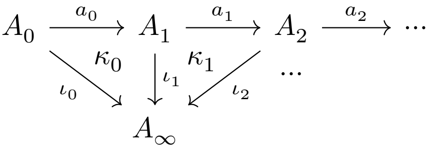


- Recall sequential colimits of diagrams $A_\bullet$



or

$$A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{\iota} A_\infty$$

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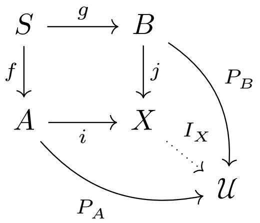

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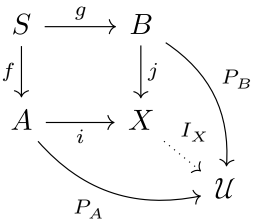

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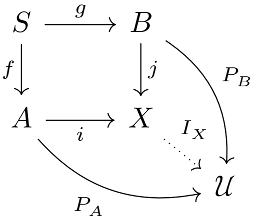
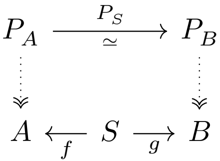
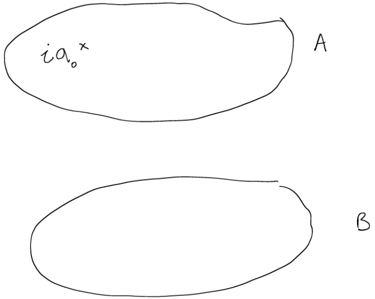
- **Descent data** (P_A, P_B, P_S): type families and equivalences $P_S(s) : P_A(fs) \simeq P_B(gs)$





Construction of the type families

- $P_A(a)$ and $P_B(b)$ defined as sequential colimits



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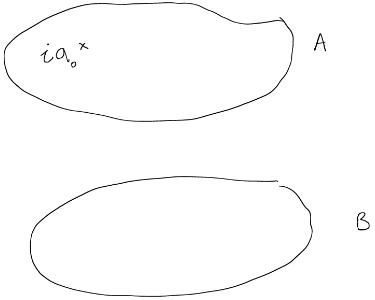

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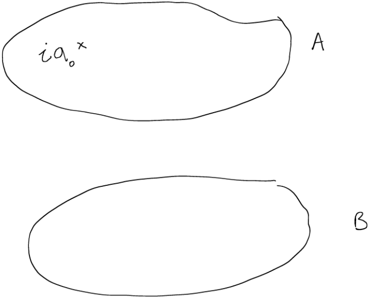

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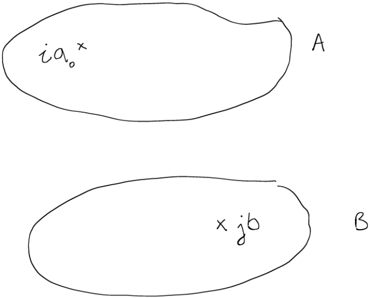

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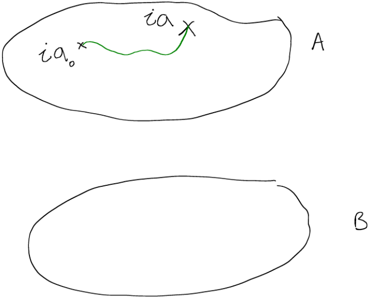

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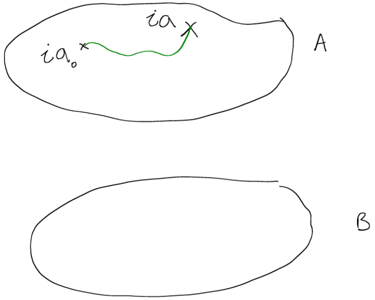

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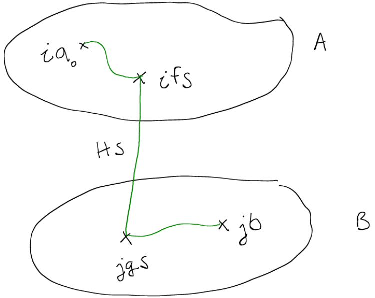

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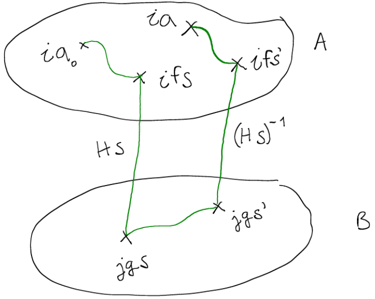

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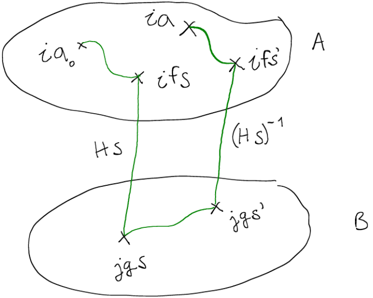
- $P_A^{n+1}(a)$ is either $P_A^n(a)$, or $P_B^{n+1}(gs)$ and a path $fs =_A a$
- "Modulo backtracking": $P_A^{n+1}(a)$ and $P_B^{n+1}(b)$ are pushouts



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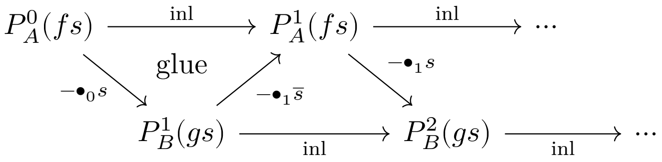

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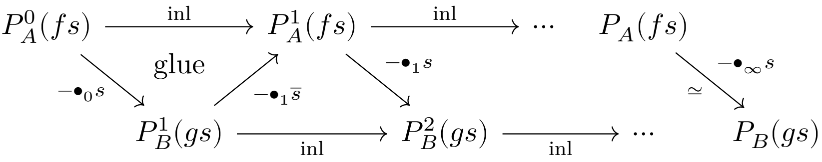

- The zigzag gives an equivalence $P_S(s) :\equiv - \bullet_\infty s$, completing (P_A, P_B, P_S), defining I_{x_0}

Identity systems

- To show $I_{x_0}(x) \simeq (x_0 = x)$, it suffices to show that I_{x_0} is an *identity system*

[3] Restatement of Kraus, von Raumer. Path Spaces of Higher Inductive Types in Homotopy Type Theory, 2019

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Definition (Induction principle of identity types of pushouts)

(P_A, P_B, P_S) with $p_0 : P_A(a_0)$ is an **identity system** if for all dependent descent data $\mathcal{Q} := (Q_A, Q_B, Q_S)$, the evaluation map $\text{ev-refl} : \text{sect}(\mathcal{Q}) \to Q_A(p_0)$ has a section.

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- Actually suffices to give a map $Q_A(p_0) \to \text{sect}(\mathcal{Q})$

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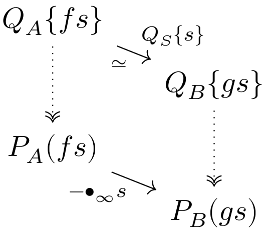
and $q_0 : Q_A(p_0)$



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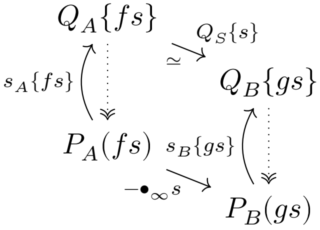
and $q_0 : Q_A(p_0)$, we need to produce

$$s_A\{a\} : (p : P_A^\infty(a)) \to Q_A(p)$$
$$s_B\{b\} : (p : P_B^\infty(b)) \to Q_B(p)$$
$$s_S\{s\} : (p : P_A^\infty(fs)) \to Q_S(s_A(p)) = s_B(p \bullet_\infty s).$$



Proof outline

- To get $s_A(a)$ and $s_B(b)$, do induction on the colimits $P_A(a)$ and $P_B(b)$: we need dependent functions

$$s_A^n\{a\} : (p : P_A^n(a)) \to Q_A(\iota_A^n(p))$$
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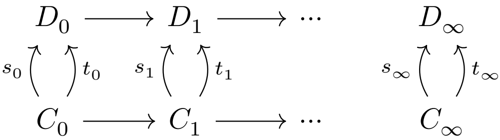
  and coherences
- The maps s^n_A and s^n_B are defined together by induction on n
- Construct $s_S\{s\}$ by proving and using functoriality theorems for sequential colimits

General results about sequential colimits

- "A homotopy of dependent diagram morphisms induces a homotopy of induced functions": $(s_\bullet \sim t_\bullet) \to (s_\infty \sim t_\infty)$



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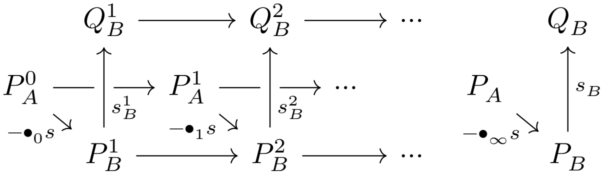

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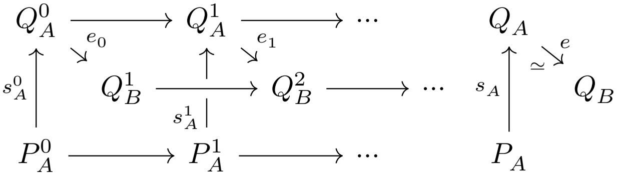

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$$
\begin{array}{ccccccccc}
P_A^0 & \longrightarrow & P_A^1 & \longrightarrow & \cdots & & P_A & & \\
& \searrow^{-\bullet_0 s} & & \searrow^{-\bullet_1 s} & & & & \searrow^{-\bullet_\infty s} & \\
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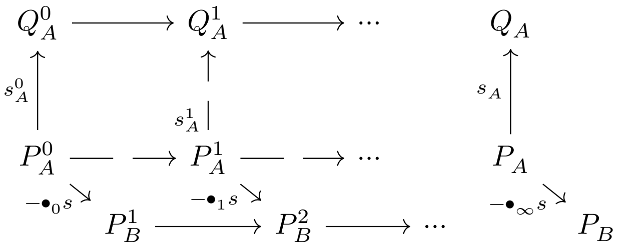

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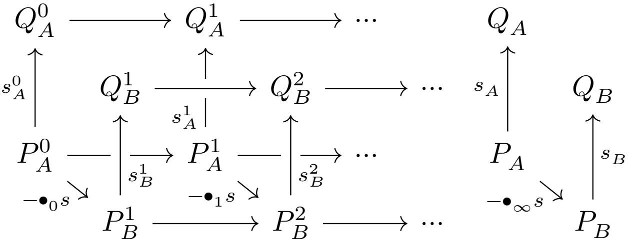

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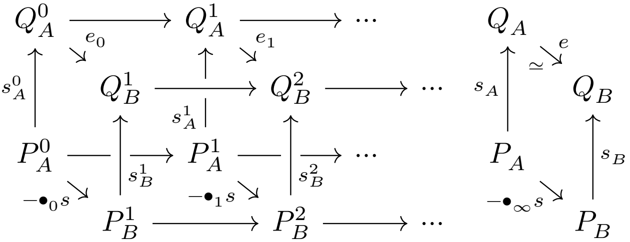

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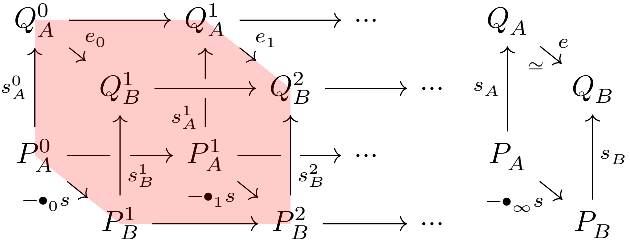

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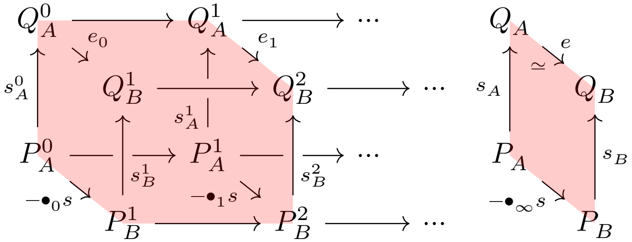

Conclusion

- The construction is formalized and proven correct
- No major issues with formalizing the construction itself
- Defining the sections s_A^n and s_B^n was more difficult, computation-wise
- Drawing the right diagrams helps finding statements of intermediate lemmas, but not so much proving them

Future work

- Formalizing applications
- Generalizing results about sequential colimits
- Optimizing: main file takes ~2.5 minutes[4], 11 GB of RAM, the rest of the library takes ~7.5 minutes; 93% spent in two definitions I didn't talk about

[4] Intel Core Ultra 7 155H

Related work

- Wärn's second article on the subject gives an "unstraightened" version of the construction
- Connors and Thorbjørnsen worked on an independent formalization in Rocq, at the time of writing the commutativity square needs to be formalized
- My master's thesis contains a more extensive description of the development of properties of various colimits leading to the formalization of the zigzag construction

Thank you for your time!

Construction of sections

- $s_A^0(\mathrm{refl}) := q_0,\ s_B^0 := \text{ex-falso}$

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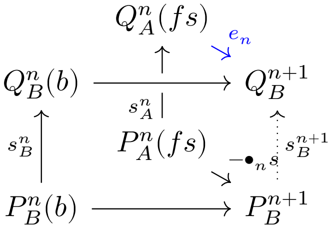

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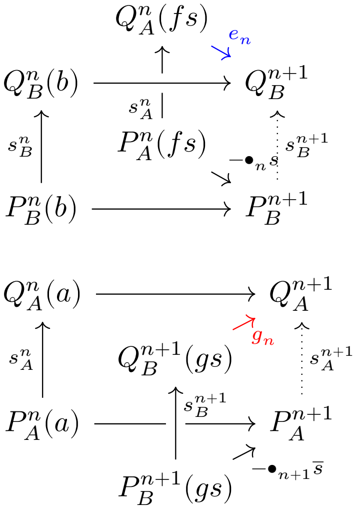

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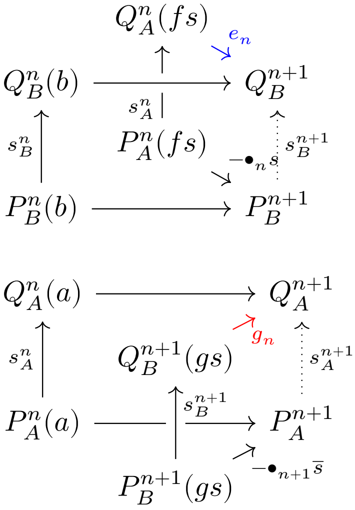
- Multiple choices for g_n



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- $s^0_A(\mathrm{refl}) := q_0$, $s^0_B :=$ ex-falso
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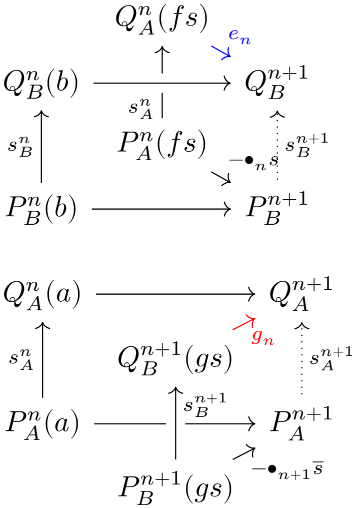
- Action of s^{n+1}_B on glue_B using $e_n \circ e_n^{-1} \sim \mathrm{id}$



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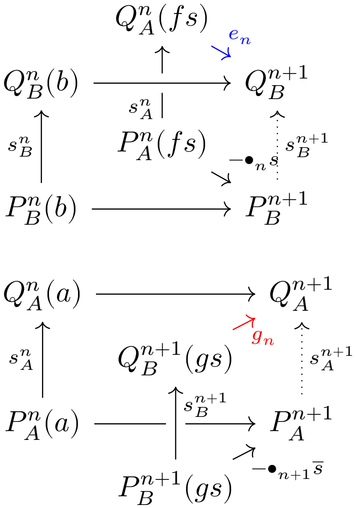
- Action of s_A^{n+1} on glue_A using the square induced by Q_S and path algebra



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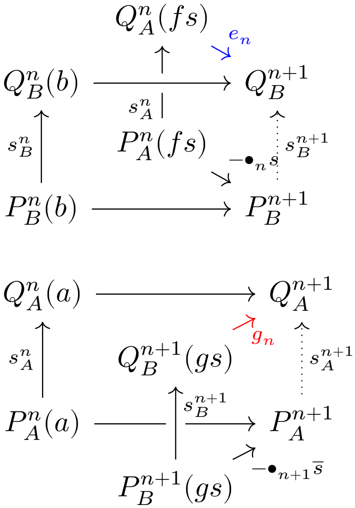
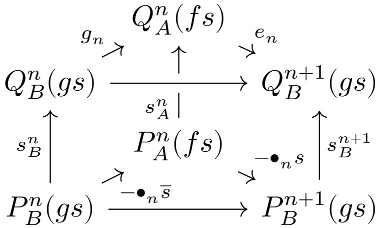
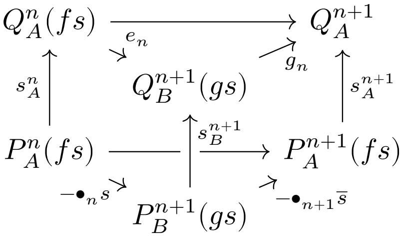
- K_A^n and K_B^n hold by computation rules for pushouts, completing s_A and s_B



Construction of the square

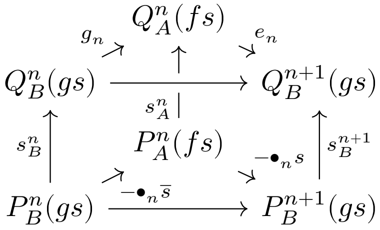
- The prisms can be filled using coherences of computation rules of pushouts

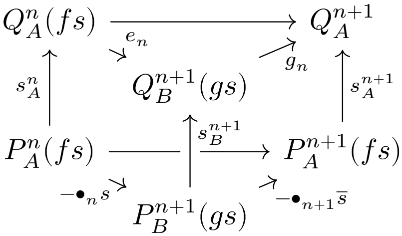




Construction of the square

- The prisms can be filled using coherences of computation rules of pushouts
- Pasting the prisms along the diagonal gives almost the correct cubes, the top face needs adjustment





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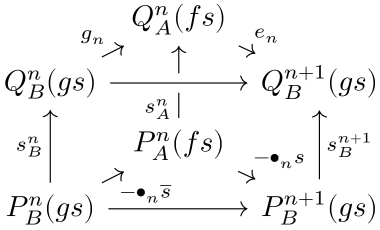

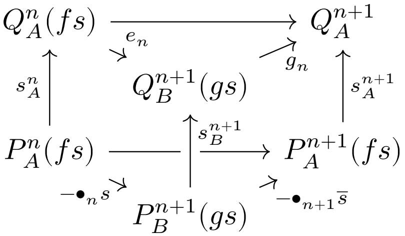

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- Proper abstraction and path induction fixes the top face
- The cubes induce the desired square s_S, which finishes the proof