Towards a formalization of Wärn's zigzag construction

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This talk presents the current state of the author's formalization of the zigzag construction due to Wärn [14]. In 2023, Wärn published a note on his construction of identity types of pushouts from sequential colimits connected by a "zigzag". This explicit construction helps concretely understand higher path spaces of pushouts. It was carried out in a general enough higher categorical setting that it is expected to be definable and provable correct in a proof assistant implementing Homotopy Type Theory [13].

However, no complete formalization has been published to date. We take this opportunity to report on an ongoing attempt by the author, which begun in his master's thesis [10]. The formalization builds on and extends the `agda-unimath` library of univalent mathematics [7], which uses the Agda proof assistant [1] with no cubical features. The intention of upstreaming the code motivates the development of reusable infrastructure for synthetic homotopy theory, as demonstrated by changes already submitted to the library, e.g. [8, 9, 11].

We present a development based on the vocabulary of descent data for pushouts, as described by Rijke [5, 6]. In particular, parameterizing constructions over pushouts by a pair of a corresponding type family and descent data, combined with the flattening lemma for pushouts due to Brunerie [2], results in a slick proof that the identity types of pushouts satisfy a slight reformulation of the universal property described by Kraus and von Raumer [4].

We show which parts of the proof of correctness of the zigzag construction needed attention during formalization in the chosen setting, which general result about sequential colimits was difficult to prove directly, and describe our experience with translating informal reasoning using "unstraightened" displayed diagrams into their "straightened" type theoretic formulations.

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In particular, the proof of correctness requires use of the following theorem:

Theorem 1 *Consider a morphism of sequential diagrams $f_\bullet : A_\bullet \to B_\bullet$, which induces a map of colimits $f_\infty : A_\infty \to B_\infty$. Then let $P : A_\infty \to \mathcal{U}$, $Q : B_\infty \to \mathcal{U}$ be type families, equipped with a family of equivalences*

$$e_\infty : (a : A_\infty) \to P(a) \simeq Q(f_\infty(a)),$$

and let $e_\bullet : P_\bullet \to_{f_\bullet} Q_\bullet$ be the induced morphism of dependent sequential diagrams over $f_\bullet$.

Provided sections $s^A_\bullet : (a : A_\bullet) \to P_\bullet(a)$ and $s^B_\bullet : (b : B_\bullet) \to Q_\bullet(b)$, and a homotopy of sections of dependent sequential diagrams $e_\bullet \circ s^A_\bullet \circ \sim s^B_\bullet \circ f_\bullet$, we obtain a commuting square $e_\infty \circ s^A_\infty \sim s^B_\infty \circ f_\infty$ in the colimit.

The theorem states that a sequence of sections of displayed squares induces a section of a displayed map in the colimit. While being graphically intuitive, this result proved to be difficult to derive directly from the dependent universal property of sequential colimits — both the presented Agda development and an independent formalization in Rocq by Connors and Thorbjørnsen [3] have been stuck on a variant of this argument.

Instead of a direct proof using the dependent universal property, we take advantage of two functoriality results. Those characterize the composites $e_\infty \circ s^A_\infty$ and $s^B_\infty \circ f_\infty$ as dependent functions induced by certain sections of dependent sequential diagrams. Homotopies of these sections turn out to be equivalent to the homotopies from the theorem, and a standard argument leveraging univalence and function extensionality shows that the homotopies of sections induce a homotopy of the induced dependent functions.

As the last step of the application of the above theorem, we need to show that the displayed squares of $e_\bullet$ are homotopic to the pasting of displayed triangles along a diagonal map $g'_n : (a : A_n) \to P_n(a) \to Q_{n+1}(g_n(a))$, where g_n is the inverse map of the zigzag construction, called $\cdot_t \overline{r}$ in the original paper. We may choose how we define g'_n, and the natural, symmetric, choice is to construct it from the map e_∞^{-1}. However, we show that a different definition is beneficial to both defining the displayed triangles, and proving that their pasting is homotopic to the desired square.

The current status of the formalization may be found on the relevant pull request page on GitHub [12].

References

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